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Causal and Systematic Relations Among Forward, Futures and Expected Spot Prices

Hun Y. Park

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Causal and Systematic Relations Among Forward, Futures and Expected Spot Prices

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Abstract

In the absence of transaction costs and in the presence of uncertainty, this paper derives implicit and explicit pricing equations of forward and futures contracts, from which causal and systematic relations can be derived among forward, futures and expected prices. The implicit pricing models of both contracts are developed by using basic microeconomic theories, i.e., the market clearing condition and the first order condition for the expected utility maximization. These models conject economic rationale for "normal backwardation" and "normal contango" processes. Adding two assumptions, lognormality and constant relative risk aversions, permits us to switch from the implicit description of a general equilibrium model to the explicit analysis of systematic patterns to the contract prices, from which empirically testable hypothesis can be derived in terms of causal relations among futures, forward and expected spot prices. Especially, it will be examined under what conditions forward and futures prices are systematically different and, at the same time, "normal backwardation" and "normal contango" are accepted as accurate descriptions of the contract prices in market equilibrium.

I. INTRODUCTION

Most recently, several papers have attributed the fundamental difference between forward and futures prices to the different payment schedule due to the property of marking-to-market in futures contracts (see Margrabe [11], Black [2], Cox, Ingersoll and Ross [4], Richard and Sundaresan [15], Jarrow and Oldfield [9], Chen and Park [3], and Park [14]). Employing different approaches, they have consistently shown that each price is the value of an asset which will pay a specific number of units of the underlying good on the maturity date.[1] Specifically, the number in the forward price is the total return from "going long" strategy in a default-free discount bond maturing at the same time as the forward contract. The number in the futures price is the total return from "rolling over" strategy in one-period bonds up to the contract maturity date. Based upon this result, the purpose of this paper is to derive implicit and explicit pricing equations of forward and futures contracts, from which causal and systematic relations can be derived among forward, futures and expected prices. Especially, it will be examined under what conditions forward and futures prices are systematically different, and, at the same time, under what conditions "Normal Backwardation" and "Normal Contango" are accepted as accurate descriptions of underlying contract prices in market equilibrium.[2] Note that "Normal Backwardation" and "Normal Contango" are referred to the processes in which contract prices (forward and futures) are systematically downward and upward biased estimates of expected spot prices over time respectively. To the author's knowledge, the explicit expressions for

the simultaneous relations among forward, futures and expected spot prices have been attracted little attentions so far. In section II, in the absence of transaction costs and in the presence of uncertainty, the implicit pricing models of forward and futures contracts are formulated through the market clearing condition and the first order condition for the expected utility maximization. Some important implications are developed from these models in terms of the economic rationale for "Normal Backwardation" and "Normal Contango." On the basis of models formulated in section II, the explicit form pricing equations of both contracts are developed in section III, adding two assumptions, log-normality and constant relative risk aversion. The simultaneous and causal relations among forward, futures and expected spot prices are derived and analyzed. Finally, section IV summarizes and concludes the paper.

II. IMPLICIT PRICING MODELS OF FORWARD AND FUTURES CONTRACTS.[3]

Notations used in this section are as follows:

Xi:	commodity X (subscript i represents the specific commodity i)
T:	maturity date of forward and futures contracts
fi(t,T):	forward price at current time t on commodity i (T > t)
Fi(t,T):	futures price at time t on commodity i
Pi(τ):	spot price of commodity i at time τ
B(t,τ):	price as of time t of a riskless bond paying one dollar at time τ(τ > t)
r(τ):	continuously compounded interest rate from time τ to time τ + 1

COV: covariance

VAR = σ^2: variance

$E_\tau(\cdot)$: expected value of the argument $(\cdot)$ at time τ

$\tilde{(\cdot)}$: randomness of the argument $(\cdot)$

As noted in the section I, the forward price is the value of a claim that pays $B(t,T)^{-1}$ units of the commodity under consideration at its maturity date T: i.e. by entering a forward contract on one unit of commodity i with a forward price fi(t,T), a person who is in the long position is paid $B(t,T)^{-1}$ units of commodity i at time T. Note however that $B(t,T)^{-1}$ is known at time t.

Once the payoff is known in terms of the number of units that can be paid at time T, a general forward contract pricing model can be developed through the market clearing condition and maximization of the expected utility function that is assumed the same for all individuals.

Consider the n+1 goods economy composed of Xo, X1, X2.... Xn, where Xo serves as a numeraire good. Investors are assumed to be rational in the sense that, under uncertainty, they are capable of finding every alternatives and choosing the best ones so as to maximize a lifetime expected utility function that is time additive.

$$E\left[\sum_{\tau=t}^{\infty} \exp(-\alpha\tau)U\{\tilde{X}o(\tau),\ \tilde{X}1(\tau)....\ \tilde{X}n(\tau)\}\right] \qquad (1)$$

where α is the utility discount factor that is known at time t and $U(\cdot)$ is a Von Neuman-morgenstern utility function that is strictly increasing, concave and twice differentiable with respect to Xi: $U\{\tilde{X}o(\tau)\cdot\tilde{X}1(\tau),...\ \tilde{X}n(\tau)\}$ is denoted by $\tilde{U}(\cdot)$ hereafter.

Suppose that two people, a buyer and a seller of forward contracts commit to β forward contracts, each contract for 1 unit of X_i at the price $f_i(t,T)$ to be paid at time T and the buyer is supposed to pay in terms of X_o. Then, the utility function at time T can be written as

$$U\{\tilde{X}_o(T) - \beta f_i(t,T)B(t,T)^{-1}, \tilde{X}_1(T), \ldots. \tilde{X}_i(T) + \beta \cdot B(t,T)^{-1}, \ldots. \tilde{X}_n(T)\} \qquad (2)$$

At the present time t, the market clearing condition and the first order condition for the expected utility maximization in a general equilibrium requires that the differentiation of the expected utility with respect to β conditional $\beta = 0$ is zero. Thus,

$$d\, E_t\left[\sum_{\tau=t}^{T-1} \exp(-\alpha\tau)\tilde{U}(T)\right] / d\beta \,\Big|_{\beta=0} = 0 \qquad (3)$$

where $E_t(\cdot)$ denotes the expected value of $(\cdot)$ at the present time t; note that equation (3) is consistent with the no arbitrage condition in that it reflects the essential point of arbitrage argument that the expected marginal utility from a forward contract should be zero taking no-initial-transfer of money into consideration.

Also, a market clearing condition simply requires that demand equals supply for each commodity, in which the market is cleared of all outstanding units of assets or contracts. Thus, even though consumers face different consumption investment opportunity sets and thus have heterogeneous perceptions on the probability distributions of risk-return on each asset or claim, it poses no problem for the determination of a market equilibrium.

From equation (2) and (3)

$$-fi(t,T)B(t,T)^{-1}Et[\exp(-(T-t)\alpha)\tilde{U}o(T)] + B(t,T)^{-1}Et[\exp(-(T-t)\alpha)\tilde{U}i(T)] = 0,$$

where $\tilde{U}i(T)$ denotes the marginal utility of commodity i at time T.

$$fi(t,T) = B(t,T)^{-1}Et[(\exp(-(T-t)\alpha)\tilde{U}i(T)]/B(t,T)^{-1}Et[(\exp(-(T-t)\alpha)\tilde{U}o(T)]$$

$$fi(t,T) = Et\tilde{U}i(T)/Et\tilde{U}o(T) \qquad (4)$$

which is the generl forward price of commodity i from the market clearing condition.[4] On the other hand, the relative spot price of commodity Xi in terms of the numeraire good Xo at each time is the ratio of marginal utility of Xi to marginal utility of Xo from the first order condition for utility maximization.

$$Pi(\tau) = Ui(\tau)/Uo(\tau) \qquad (5)$$

Thus,

$$Et\{\tilde{P}i(T)\} = Et\{\tilde{U}i(T)/\tilde{U}o(T)\} \qquad (6)$$

Note from equations (4) and (6) that the question whether the forward price follows the "Normal backwardation" process (Keynes [10], Hicks [7], Houthakker [8]) or "Normal Contango" (Telser [17]) or whether the forward price is equal to the expected spot price depends on whether $Et\tilde{U}i(T)/Et\tilde{U}o(T)$ is less or greater than, or equal to $Et\{Ui(T)/Uo(T)\}$.

Also, following this procedure, it is very simple to derive the relation between the forward price and the expected future spot price that is somewhat similar to the results that Richard and Sundaresan [15] obtained through a fairly complicated procedure.

$$
\begin{aligned}
fi(t,T) &= Et\tilde{U}i(T)/Et\tilde{U}o(T) \\
&= Et\{\tilde{U}o(T)\cdot\tilde{P}i(T)\}/Et\tilde{U}o(T) \qquad \text{From equation (1)} \\
&= [COVt\{\tilde{U}o(T),\tilde{P}i(T)\} + Et\tilde{U}o(T)\cdot Et\tilde{P}i(T)]/Et\tilde{U}o(T) \\
&= Et\tilde{P}i(T) + COVt\{\tilde{U}o(T),\tilde{P}i(T)\}/Et\tilde{U}o(T) \qquad (7)
\end{aligned}
$$

From equation (7), the relation between the forward price and the expected spot price of commodity i in connection with the controversy of "Normal Backwardation" or "Normal Contango" can be described as follows: if the correlation between the future spot price and the marginal utility of a numeraire good at the maturity date is positive or negative, the forward price is greater or less than the expected spot price respectively. If there is no correlation between the spot price and the marginal utility of a numeraire good at time T, the forward price will be equal to the expected spot price at the maturity date as Black [2], Dusak [5], and Samuelson [16] argue.

These results are intuitively appealing in the following sense: for example, suppose that the covariance between the spot price and the marginal utility of a numeraire (it may be money in dollar terms or one-period discount bond price as will be assumed later in this paper) at the maturity date T is negative. Considering a long position in a forward contract, the spot price, $\tilde{P}i(T)$, has a positive correlation with the payoff, and thus with the return at time T. Then, the negative covariance between returns or profits and the marginal utility of money or one-period bond price implies decreasing marginal utility and thus that the magnitude of the marginal utility of return or profit is less than that of the marginal utility of loss with the same dollar amount.

This result reflects the notion of risk aversion of investors. Thus, in order to induce risk averse investors to commit to a forward contract, there must be compensation for bearing this risk in the form of a positive return. This logic leads in turn to the conclusion that the forward price should be less than the expected spot price, which conforms with the Keynes-Hicks-Houthakker's argument, the "Normal Backwardation."

It is very important to note that the expected spot price is expressed in terms of relative prices and that they are explicitly incorporated into pricing of the forward contract. This is plausible when we take into account the fact that consumption bundles of each investor depend mainly on relative prices, but not on absolute prices.

On the same line of logic, futures contract pricing models can be formulated. As discussed in the introduction, the futures price is the value of a claim that pays $\exp \sum_{\tau=t}^{T-1} r(\tau)$ units of the commodity under consideration at time T.

Note that $\exp \sum_{\tau=t}^{T-1} r(\tau)$ units of the commodity is unknown at time t, and thus engaging in futures contract is a speculation not only on the future spot price but on the number of units of commodities than can be paid for the futures contract at the maturity date.[5] From the same logic that was employed in the forward contract pricing model, if a person engages in β units of futures contracts, each unit of the contract for 1 unit of Xi at the price Fi(t,T) by promising to pay for it with Xo, then his utility function at time T is

$$U\{\tilde{X}o(T)-\beta\cdot Fi(t,T)\exp\sum_{\tau=t}^{T-1} r(\tau),\tilde{X}1(T),\tilde{X}2(T),.... \tilde{X}i(T)+\beta\cdot\exp\sum_{\tau=t}^{T-1} r(\tau),.... \tilde{X}n(T)\} \quad (8)$$

From equation (3),

$$-Fi(t,T)Et\{\sum_{\tau=t}^{T-1}\exp(-\alpha\tau)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\tilde{U}o(T)\}+Et\{\sum_{\tau=t}^{T-1}\exp(-\alpha\tau)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\tilde{U}i(T)\} = 0$$

Thus,

$$Fi(t,T) = Et\exp\{\sum_{\tau=t}^{T-1}(r(\tau)-\alpha)\cdot\tilde{U}i(T)\}/Et\exp\{\sum_{\tau=t}^{T-1}(r(\tau)-\alpha)\cdot\tilde{U}o(T)\} \quad (9)$$

which is the general futures contract pricing equation that is deduced from the market clearing condition.

Besides, the relation between the expected spot price and the futures price can be easily derived from equation (9) as follows (See foctnote 6 for proof)

$$Fi(t,T) = Et\,\tilde{P}i(T) + COV\{\tilde{P}i(T),\tilde{U}o(T)\exp\sum_{\tau=1}^{T-1} r(\tau)\}/Et\,\tilde{U}o(T)\exp\sum_{\tau=1}^{T-1} r(\tau) \quad (10)$$

It is obvious now that the relation between the futures price and the expected spot price depends on the covariance between the spot price and the marginal utility of a numeraire good multiplied by $\exp\sum_{\tau=t}^{T-1} r(\tau)$ at the maturity date. It depends no longer on the simple covariance between the spot price and the marginal utility of a numeraire good due to the possibility of stochastic one-period interest rates.

It is important to note that the market efficiency concept does not say anything about equations (7) and (10). In other words, the question whether the futures and forward prices are greater than, or equal to, or less than the expected spot price is irrelevant to the question whether or not those markets, forward and futures, are efficient in processing information.

III. RELATIONS AMONG FORWARD, FUTURES AND EXPECTED PRICES

3.1 Explicit Form Pricing Equations

In order to derive empirically testable hypotheses and their implications, we assume that the joint distribution of n+1 commodities is multivariate lognormal, which implies that the joint distribution of any two commodities is bivariate lognormal; in fact, we need means and variances only for Xi and Xo and their covariances assuming that a buyer of forward or futures contract on Xi pays for it in terms of Xo.

If Xi is lognormally distributed, then log Xi is normally distributed, and the density of the lognormal distribution is given by

$$F(Xi;\mu,\sigma^2) = (1/Xi\sqrt{2\pi\sigma})\exp\{-1/2\sigma^2(\log Xi-\mu)^2\}$$

where $E(Xi) = \exp(\mu+1/2\cdot\sigma^2)$, and $Var(Xi) = \exp(2\mu+2\sigma^2) - \exp(2\mu+\sigma^2)$

$$E(\log Xi) = \mu \text{ and } Var(\log Xi) = \sigma^2 \qquad (11)$$

In addition to this lognormality assumption of commodity distribution, we assume that the consumption-investment decision of each individual on n+1 commodities is based on a power utility function such as $(k/1-\gamma)X^{1-\gamma}$, where the parameter showing the degree of risk aversion

is positive and characterized by decreasing absolute risk aversion and constant relative risk aversion.[7]

Notation used in this section is as follows:

$\sigma^2 i(\tau)$: variance of $\log \tilde{X}i(\tau)$ at time τ.

$\sigma^2 o(\tau)$: variance of $\log \tilde{X}o(\tau)$ at time τ.

$\sigma^2 r(\tau)$: variance of $\sum_{\tau=t}^{T-1} r(\tau)$.

ρio: correlation between $\log \tilde{X}i(\tau)$ and $\log \tilde{X}o(\tau)$.

ρir: correlation between $\log \tilde{U}i(\tau)$ and $\sum_{\tau=t}^{T-1} r(\tau)$.

ρor: correlation between $\log \tilde{U}o(\tau)$ and $\sum_{\tau=t}^{T-1} r(\tau)$.

$$\tilde{\phi}(\tau) = \log \tilde{U}i(\tau)$$

$$\tilde{\Phi}(\tau) = \log \tilde{U}o(\tau)$$

$$\tilde{\phi}^1(\tau) = \log \tilde{U}i(\tau)\exp \sum_{\tau=t}^{T-1} r(\tau) = \log \tilde{U}i(\tau) + \sum_{\tau=t}^{T-1} r(\tau)$$

$$\tilde{\Phi}^1(\tau) = \log \tilde{U}o(\tau)\exp \sum_{\tau=t}^{T-1} r(\tau) = \log \tilde{U}o(\tau) + \sum_{\tau=t}^{T-1} r(\tau)$$

$$\tilde{Z}(\tau) = \tilde{\phi}(\tau) - \tilde{\Phi}(\tau) = \tilde{\phi}^1(\tau) - \tilde{\Phi}^1(\tau).$$

From the assumption of power utility function,

$$Ut\{\tilde{X}i(T), \tilde{X}o(T)\} = (1/(1-a))\tilde{X}i(T)^{1-a} + (1/(1-b))\tilde{X}o(T)^{1-b} \qquad (12)$$

where positive a and b show the degree of risk aversion on commodity Xi and Xo respectively.[8]

$$\tilde{U}i(T) = \tilde{X}i(T)^{-a}, \quad \tilde{U}o(T) = \tilde{X}o(T)^{-b}$$

$$\tilde{\phi}(T) = \log \tilde{U}i(T) = -a \log \tilde{X}i(T)$$

$$\tilde{\Phi}(T) = \log \tilde{U}o(T) = -b \log \tilde{X}o(T)$$

$$\tilde{Z}(T) = \tilde{\phi}(T) - \tilde{\Phi}(T) = \log\{\tilde{U}i(T)/\tilde{U}o(T)\} = \log \tilde{P}i(T)$$

$$Et\tilde{Z}(T) = Et\tilde{\phi}(T) - Et\tilde{\Phi}(T)$$

$$\text{Var } \tilde{\phi}(T) = \text{Var}\{\log \tilde{U}i(T)\} = a^2 \cdot \sigma^2 i(T) \qquad (13)$$

$$\text{Var } \tilde{\Phi}(T) = \text{Var}\{\log \tilde{U}o(T)\} = b^2 \cdot \sigma^2 o(T)$$

$$\text{Var } \tilde{\phi}^1(T) = \text{Var}\{\log \tilde{U}i(T) + \sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= a^2 \cdot \sigma^2 i(T) + \sigma^2 r(T) + 2\rho ir\ a\ \sigma i(T)\sigma r(T)$$

$$\text{Var } \tilde{\phi}^{-1}(T) = \text{Var}\{\log \tilde{U}o(T) + \sum_{\tau=t}^{T-1} r(T)\}$$

$$= b^2 \cdot \sigma^2 o(T) + \sigma^2 r(T) + 2\rho or\ b\sigma o(T)\sigma r(T)$$

From equations (11) and (13),

$$fi(t,T) = Et\tilde{U}i(T)/Et\tilde{U}o(T)$$

$$= \exp\{Et\tilde{\phi}(T) + 1/2a^2 \cdot \sigma^2 i(T)\}/\exp\{Et\tilde{\Phi}(T) + 1/2b^2 \cdot \sigma^2 o(T)\}$$

$$= \exp\{Et\tilde{Z}(T) + 1/2(a^2 \cdot \sigma^2 i(T) - b^2 \cdot \sigma^2 o(T))\} \qquad (14)$$

$$Fi(t,T) = Et\{\tilde{U}i(T) \cdot \exp \sum_{\tau=t}^{T-1} r(\tau)\}/Et\{\tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= \exp\{E\tilde{\phi}^1(T) + 1/2(a^2 \cdot \sigma^2 i(T) + \sigma^2 r(T) + 2\rho ir\ a\sigma i(T)\sigma r(T))\}/$$

$$\exp\{E\tilde{\Phi}^1(T) + 1/2(b^2\sigma^2 o(T) + \sigma^2 r(T) + 2\rho or\ b\sigma o(T)\sigma r(T))\}$$

$$= \exp\{Et\tilde{Z}(T) + 1/2((a^2\sigma^2 i(T) + 2\sigma r(\rho ir\ a\sigma i(T) - \rho or\ b\sigma o(T))$$

$$-b^2\sigma^2 o(T))\} \qquad (15)$$

Also,

$$Et\tilde{P}i(T) = Et\{\tilde{U}i(T)/\tilde{U}o(T)\} = Et\{\exp\tilde{\phi}(T)/\exp\tilde{\Phi}(T)\} = Et\{\exp(\tilde{\phi}(T) - \tilde{\Phi}(T)\}$$

$$\text{Var}\{\log \tilde{P}i(T)\} = \text{Var}\{\tilde{\phi}(T) - \tilde{\Phi}(T)\}$$

$$= a^2 \cdot \sigma^2 i(T) + b^2 \cdot \sigma^2 o(T) - 2\rho io\ ab\sigma i(T)\sigma o(T)$$

(16)

Thus, from (11), (13), and (16)

$$E_t\tilde{P}_i(T) = \exp\left[E_t\tilde{Z}(T) + 1/2\{a^2\cdot\sigma^2 i(T) + b^2\cdot\sigma^2 o(T) - 2\rho io\ ab\sigma i(T)\sigma o(T)\}\right] \tag{17}$$

Equations (14), (15), and (17) show the explicit form pricing models for forward contract, futures contract and the expected future spot price under the given assumptions.

The most important implication from equation (14), (15), and (17) is that we can not only specify the causal relations among those three prices, but also derive testable hypotheses. In other words, by examining the variables in the equations, we can investigate under what conditions a "Normal Backwardation" or a "Normal Contango" process of forward or futures prices is possible and under what conditions the forward contract price is equivalent to the futures contract price.

3.2 Relation Between Forward and Futures Prices

From equation (14) and (15),

$$\log F_i(t,T) - \log f_i(t,T) = \sigma r\{\rho ir\ a\sigma i(T) - \rho or\ b\sigma o(T)\} \tag{18}$$

Then, the following implications follow directly from equation (18),

1) if $\sigma r = 0$, then $F_i(t,T) = f_i(t,T)$.

If the one-period interest rate is nonstochastic, a futures contract is equivalent to a forward contract regardless of the degree of risk aversion and the variance of each commodity. This is intuitively plausible in that a nonstochastic interest rate implies that the hedge ratio for both futures and forward contract is given at time t when the

contracts are open, and thus the contracts should be equivalent under rational expectations. However, note that zero variance of the one period interest rate is sufficient but not necessary for the equivalence of forward and futures contracts.

2) If the variance of the interest rate is non-zero, the question of whether Fi(t,T) is greater than or equal to or less than fi(t,T) depends on the magnitude of $\rho_{ir}\,\sigma_{ci}(T)$ and $\rho_{or}\,b\sigma_o(T)$. Assuming a one-period riskfree discount bond as the numeraire good, following implications can be deduced. Hereafter, in this paper, the numeraire good is assumed to be a one-period riskfree discount bond.[9]

Then, $\sigma_o(T) = \sigma_r(T)$; note that $\text{Var} \sum_{\tau=t}^{T-1} r(\tau) = \sum_{\tau=t}^{T-1} \text{Var}\, r(\tau)$ if interest rate at period τ is independent of that at period $\tau + 1$.

Also, noting that the bond price is negatively correlated with the interest rate, the following causal relations can be derived:

$$\text{If } ab\,\rho_{io}\,\sigma_{ci}(T)\,\sigma_o(T) \lessgtr b^2 \cdot \sigma^2_o(T),$$

$$\text{then } Fi(t,T) \gtrless fi(t,T)$$

This implies that if the covariance between the price of the commodity i and the price of one-period riskfree discount bond is less than the variance of the price of the bond, the futures price is greater than the forward price. This result is intuitively plausible in that futures prices depend on the correlation of spot prices and interest rates, while forward prices do not.[10]

Generally, if interest rates go up, it becomes more costly for speculators to buy commodities and for firms to build up inventories, thus increasing commodity prices. Considering the negative correlation between interest rates and the bond prices, the covariance between bond

prices and storable commodity prices tends to be negative, thus the futures price tends to be greater than the forward price. On the other hand, financial futures such as Treasury bills, are expected to have a high correlation with the one-period bond prices, so that futures prices for Treasury bills tend to be lower than their forward prices.

3.3 The Issue of Normal Backwardation or Normal Contango

By comparing equation (14) and (15) with equation (17), we can specify explicitly under what conditions, the forward or futures price is an unbiased estimate of the expected future spot price and when they are supposed to follow the "Normal Backwardation" or Normal Contango" process.

Subtracting equation (17) from (14) and (15) respectively after taking logarithm

$$\log fi(t,T) - \log EtPi(T) = b\sigma o(T)\{\rho io\ a\sigma i(T) - b\sigma o(T)\} \tag{19}$$

$$\begin{aligned}\log Fi(t,T) - \log EtPi(T) &= -b^2 \cdot \sigma^2 o(T) + \sigma r\{\rho ir\ a\sigma i(T) \\ &\quad -\rho or\ b\sigma o(T)\} + \rho io\ ab\sigma i(T)\sigma o(T) \\ &= \sigma o\{a\ \rho io\ \sigma i(T) - b\sigma o(T)\}(b-1)\end{aligned} \tag{20}$$

assuming the one-period discount bond as a numeraire good; note that if we subtract equation (19) from equation (20), it turns out to be equal to equation (18).

From equations (19) and (20), following implications can be derived immediately.

1) First, if the interest rate is nonstochastic and thus $\sigma o(T) = 0$, then $fi(t,T) = Fi(t,T) = Et\tilde{P}i(T)$; both forward and futures contract prices are unbiased estimates of future expected spot prices. This has intuitive explanation because of the reasons stated in section 3.1. Note however that this is not a necessary condition for the equivalence of forward and futures contract, nor for them to be unbiased estimates of the expected spot price.

2) Second, assuming that $b > 1$, which is the case of the generalized negative power utility function, a fairly striking result follows from the equations (18), (19), and (20); the comparison of futures prices or forward prices with expected spot prices is equivalent to that of forward prices with futures prices. In other words, the simultaneous relations among those three prices hold as follows:

$$\text{If } ab\,\rho io\,\sigma i(T)\,\sigma o(T) \lessgtr b^2 \cdot \sigma^2 o(T),$$

$$\text{then } EP(T) \gtreqless F(t,T) \gtreqless f(t,T) \qquad (21)$$

In words, if the covariance between the price of commodity i and the discount bond price is less than the variance of the bond price, both prices, futures and forward prices, will follow a "Normal Backwardation" process and if the former is greater than the latter, both prices will follow a "Normal Contango" process.

Concerning the possibility of hedging instrument of the contracts, if commodity i and the one-period discount bond are negatively correlated, so that the commodity is a good candidate for a hedging instrument against changes in the price of the one-period discount bond, both

futures and forward price are downward biased estimates of the expected spot price. "Normal Backwardation" is a natural deduction for the commodity.

IV. SUMMARY AND CONCLUSION

The fact that we can express the payoff of forward and futures contracts in terms of the number of units of commodities is a fairly important result, which gives us a good deal to go on and from which we can deduce some interesting implications.

Based on the number of units of commodities that can be paid for a futures and a forward contracts, this paper developed implicit pricing models of both contracts through the market clearing condition and the first order condition for the expected utility maximization. These models conject economic rationale for "Normal Backwardation" and "Normal Contango."

Adding two assumptions on the underlying process of assets (log-normality) and the utility function (constant relative risk aversions) permitted us to switch from the implicit description of a general equilibrium model to the explicit analysis of systematic patterns to the two contract prices, from which empirically testable hypothesis could be derived in terms of causal relations among futures, forward and expected spot prices.

The following implications were immediate; if the interest rate is nonstochastic, futures contracts are equivalent to forward contracts regardless of the degree of risk aversion and the variance of commodities, and at the same time, both contract prices are unbiased estimates of

future expected spot prices. If the covariance between the changes in commodity prices and the changes in discount bond prices is less than the variance of the changes in bond prices, futures prices are greater than forward prices, and simultaneously both of these contract prices are downward biased estimates of the expected spot prices, so that the "Normal Backwardation" is a natural deduction for describing the contract prices. If the covariance is equal to or greater than the variance, futures prices are equal to or less than forward prices respectively, and at the same time, both contract prices are unbiased, or upward biased estimates of expected spot prices correspondingly. This implies that if a commodity provides a hedging instrument against changes in bond prices, "Normal Backwardation" process can be said to be an accurate description of both futures and forward prices.

In the process, it was clearly confirmed that the systematic difference between futures and forward prices, and "Normal Backwardation" or "Normal Congango" are not inconsistent with market equilibrium, or market efficiency. Nevertheless, since the models are formulated in a simplified economy, the analysis in this paper can never be perfect in explaining the sources of deviations from the models. The purpose of this paper, however, is not so much to introduce all the many factors that can theoretically influence futures and forward prices simultaneously into one equation as it is to find the best explanation for the causal relations among futures, forward and expected spot prices in the simple economy.

Footnotes

[1]For descriptions of fundamental differences between futures and forward contracts, see Black [2], Cox, Ingersoll and Ross [4], Margrobe [11], Jarrow and Oldfield [9], Richard and Sundaresan [15], Chen and Park [3], and Park [14].

[2]There is a slight difference between "Normal Backwardation" and "Backwardation." "Normal Backwardation" refers to the situation where the expected future spot price is greater than the contract price while "Backwardation" refers to the situation where the current spot price is greater than the contract price. The terms "Normal Contango" and "Contango" are the reverse respectively. This clarification of terminologies is due to Professor J. H. McCulloch. See Keynes [10], Hicks [7], Houthakker [8], Arrow [1], McCulloch [12] for normal backwardation, and Hardy [6] and Telser [17] for normal contango.

[3]Throughout this paper, the consideration of margin requirement is ruled out; it is important to note that margin requirements are not partial equity payments against the market value of the commodity represented by the contract, as it is when buying common stocks, but a guarantee in the event of adverse price movements.

Nevertheless, a controversy over margin requirements exists in connection with investors' optimal portfolio construction. For example, Telser [18] argues that margin requirement should be incorporated in the pricing equations because they may disrupt individual investor's optimal portfolio allocation and thus induce costs even though they are in the form of interest-earning securities like Treasury bills. However, as long as individual investors can borrow in a perfect capital market against their portfolios to buy Treasury bills for the purpose of posting them as margin requirements, it doesn't induce any cost. In other words, the opportunity cost for posting margin requirements is zero, assuming no transaction costs in a perfect capital market.

[4]J. H. McCulloch [13] derived the same pricing equation in connection with short-lived options pricing when the underlying distribution of price is log-symmetric stable.

[5]This was pointed out also by Cox, Ingersoll and Ross [4], and Richard and Sundaresan [15].

[6] $$F_i(t,T) = E_t\{\tilde{U}_i(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\}/E_t\{\tilde{U}_o(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)$$

since α is deterministic

$$= E_t\{\tilde{P}_i(T)\cdot\tilde{U}_o(T)\cdot\exp\sum_{\tau=t}^{T-1} r(\tau)\}/E_t\{\tilde{U}_o(T)\exp\sum_{\tau=t}^{T-1} r(\tau)\}$$

from equation (1)

$$= \left[\mathrm{COV}\{\tilde{P}i(T), \tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\} + Et\, \tilde{P}i(T) \cdot Et\, \tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau) \right] / Et\{\tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\}$$

$$= Et\, \tilde{P}i(T) + \mathrm{COV}\{\tilde{P}i(T), \tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\} / Et\{\tilde{U}o(T) \exp \sum_{\tau=t}^{T-1} r(\tau)\}$$

which is the equation (10). This equation (10) is quite similar to the models of Richard and Sundaresan [15] obtained from a quite different approach.

[7] If $U(X) = (k/1-\gamma)X^{1-\gamma}$, $U^1 = X^{-\gamma}$, $U^{11} = -k/\gamma\, X^{-\gamma-1}$

Thus, the absolute risk aversion,

$$R_A(X) = -U^{11}/U^1 = -k/\gamma\, X^{-\gamma-1}/X^{-\gamma} = k/\gamma\, X^{-1}$$

$$R_A{}^1(X) = -k/\gamma\, X^2 < 0,$$

which implies the decreasing absolute risk aversion.

Also, the relative risk aversion,

$$R_R(X) = R_A(X)X = k/\gamma\, X^{-1} X = k/\gamma$$

$$R_R{}^1(X) = 0,$$

which implies the constant relative risk aversion.

[8] The general power utility function is given by $(Ki/1-\gamma)Xi^{1-\gamma}$. In this paper, Ki is assumed to equal one for simplicity with no loss of generality for comparison between forward and futures prices and expected spot prices.

[9] The choice of a discount bond is consistent with an arbitrage argument in a complete market. Suppose that there is a futures contract in every state of the world. In the other words, a futures contract is an asset with a payoff in the next period that is equal to the state price that is uncertain, assuming the existence of a futures contract on each state that expires at every instant and another created that matures in the next instant. Then it is well known that a set of futures contract plus a risk free one-period discount bond can achieve the complete market in Arrow and Debrew sense.

In the light of the above reasoning, the choice of the one-period discount bond is believed to be a reasonable choice as a numeraire good.

[10] This was indicated by Cox, Ingersoll and Ross [4].

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